Potential Wadge classes

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Abstract. Let $\mathbf{\Gamma}$ be a Borel class, or a Wadge class of Borel sets, and $2\leq d\leq\omega$ a cardinal. We study the Borel subsets of $\mathbb{R}^d$ that can be made $\mathbf{\Gamma}$ by refining the Polish topology on the real line. These sets are called potentially $\mathbf{\Gamma}$. We give a Hurewicz-like test to recognize potentially $\mathbf{\Gamma}$ sets. The method gives a new approach to the study of Wadge classes, and another instance of an effective property at the origin of many dichotomy results in descriptive set theory.

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The title shows that we will study the Borel subsets of products of Polish spaces. In fact, the method that we will see gives a new approach to the study of Wadge classes in the general context, and not only in products. This work also goes in the direction of finding common points to many dichotomy results in descriptive set theory concerning equivalence relations, quasi-orders and arbitrary analytic sets. More specifically, in many such dichotomy results, an Σ^1_1 set plays a crucial role: its emptiness is equivalent to the first possibility of the dichotomy. Let us give an example, concerning the Harrington-Kechris-Louveau dichotomy:

Theorem 1 *Let X be a recursively presented Polish space, E a Δ^1_1 equivalence on X. Then exactly one of the following holds :*

(a) E is smooth.

(b) There is a continuous embedding from $\mathbb{E}_0$ into E.

The Σ^1_1 set is $\overline{E}^{\Sigma^2_X}\setminus E$, where Σ_X is the Gandy-Harrington topology on X.

We will give the Σ^1_1 set corresponding to the dichotomy we are interested in, for some significative example. We first recall the basic definitions. In the sequel, $\mathbf{\Gamma}$ will be a class of Borel subsets of 0-dimensional Polish spaces.

Definition 2 *We say that $\mathbf{\Gamma}$ is a $Wadge\ class\ of\ Borel\ sets$ if we can find $A_0\in\mathbf{\Delta}^1_1(\omega^\omega)$ such that $\mathbf{\Gamma}=\{f^{-1}(A_0)\mid f\ \text{ continuous}\}$.*

We work in 0-dimensional Polish spaces to ensure the existence of enough continuous functions. This will not be a real restriction for us, since we will work up to finer Polish topologies. The Wadge hierarchy (i.e., the inclusion of classes) is the finest hierarchy of topological complexity considered in descriptive set theory. It extends the usual hierarchies of Borel (with $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$) and Lavrentieff (with transfinite differences of $\mathbf{\Sigma}^0_\xi$ sets) classes.

We now specify the notion of complexity we consider in products. It was introduced by Louveau in dimension two. Here we work with any dimension d making sense in the context of classical descriptive set theory. In particular, d will be a cardinal with $2\leq d\leq\omega$ since 2^{ω_1} is not metrizable.

Definition 3 *Let $(X_i)_{i\in d}$ be a sequence of Polish spaces, B a Borel subset of $\Pi_{i\in d}\ X_i$. We say that B is $potentially\ in\ \mathbf{\Gamma}$ $\big($denoted $B\in pot(\mathbf{\Gamma})\big)$ if, for each $i\in d$, there is a finer 0-dimensional Polish topology τ_i on X_i such that $B\in\mathbf{\Gamma}\big(\Pi_{i\in d}\ (X_i,\tau_i)\big)$.*

One should emphasize the fact that the point of this definition is to consider product topologies. The notion of potential complexity is an invariant for the usual quasi-order $\leq_B$ used to compare analytic equivalence relations, in the sense that if $(X,E)\ \leq_B\ (Y,F)$ and F is pot($\mathbf{\Gamma}$), then E is pot($\mathbf{\Gamma}$) too. Our main result is the following:

Theorem 4 *Let $\mathbf{\Gamma}$ be a Wadge class of Borel sets, or the class $\mathbf{\Delta}^0_\xi$ for some $1\leq\xi<\omega_1$. Then there are Borel subsets $\mathbb{S}_0$, $\mathbb{S}_1$ of $(d^\omega)^d$ such that for any sequence of Polish spaces $(X_i)_{i\in d}$, and for any disjoint analytic subsets A_0, A_1 of $\Pi_{i\in d}\ X_i$, exactly one of the following holds:*

(a) The set A_0 is separable from A_1 by a pot($\mathbf{\Gamma}$) set.

(b) For each $i\in d$ there is $f_i:d^\omega\rightarrow X_i$ continuous such that $\mathbb{S}_\varepsilon\subseteq(\Pi_{i\in d}\ f_i)^{-1}(A_\varepsilon)$ for each $\varepsilon\in 2$.

This result extends the Debs-Lecomte dichotomy about potentially $\mathbf{\Pi}^0_\xi$ subsets of a product of two Polish spaces. It generalizes to products the Louveau-Saint Raymond result, which itself generalized Hurewicz's theorem characterizing G_δ sets. The extension of the Debs-Lecomte dichotomy goes in three directions: it extends to

- Any dimension d.
- The self-dual Borel classes $\mathbf{\Delta}^0_\xi$.
- Any Wadge class of Borel sets, which is the hardest part.

The motivation for doing this is not only a will to give a simple generalization. The dichotomy is a theorem of continuous reduction, and the notion of a Wadge class is also about continuous reductions. So part of the conclusion of the theorem and the definition of a Wadge class are very similar.

We set $\check{\mathbf{\Gamma}}:=\{\neg A\mid A\in\mathbf{\Gamma}\}$, and say that $\mathbf{\Gamma}$ is $self$-$dual$ if $\mathbf{\Gamma}=\check{\mathbf{\Gamma}}$. Note that we can have $\mathbb{S}_0\cup\mathbb{S}_1$ closed if $\mathbf{\Gamma}$ is not self-dual, so that the reduction holds on a closed set. We now specify the notions of smallness of the closed set ensuring the possibility of a reduction.

Notation. If $\mathcal{X}$ is a set, then $\vec{x}:=(x_i)_{i\in d}$ is an arbitrary element of $\mathcal{X}^d$. If $\mathcal{T}\subseteq\mathcal{X}^d$, then we denote by $G^{\mathcal{T}}$ the graph with set of vertices $\mathcal{T}$, and with set of edges

$$\Big\{\{\vec{x},\vec{y}\}\subseteq\mathcal{T}\mid\vec{x}\neq\vec{y}\text{ and }\exists i\in d\ \ x_i=y_i\Big\}.$$

Definition 5 *(a) We say that $\mathcal{T}$ is one-$sided$ if the following holds:*

$$\forall\vec{x},\vec{y}\in\mathcal{T}\ \ \forall i,j\in d\ \ \big(\ (\vec{x}\neq\vec{y}\ \text{ and }\ x_i=y_i\ \text{ and }\ x_j=y_j)\ \Rightarrow\ i=j\ \big).$$

(b) We say that $\mathcal{T}$ is $almost\ acyclic$ if for every $G^{\mathcal{T}}$-cycle $(\overrightarrow{x^n})_{n\leq L}$ there are $i\in d$ and $k<m<n<L$ such that $x^k_i=x^m_i=x^n_i$.

(c) We say that a tree T on d^d is a $tree\ with\ suitable\ levels$ if, for each integer l, the set defined by $\mathcal{T}^l:=T\cap(d^l)^d\subseteq(d^l)^d\equiv(d^d)^l$ is finite, one-sided and almost acyclic.

Now we state the main result in two parts for non self-dual classes:

Theorem 6 *There are a tree T_d with suitable levels, together with, for each non self-dual Wadge class of Borel sets $\mathbf{\Gamma}$, $\mathbb{S}^d_{\mathbf{\Gamma}}\in\mathbf{\Gamma}(\lceil T_d\rceil)$ which is not separable from $\lceil T_d\rceil\setminus\mathbb{S}^d_{\mathbf{\Gamma}}$ by a pot($\check{\mathbf{\Gamma}}$) set.*

Theorem 7 *Let T_d be a tree with suitable levels, $\mathbf{\Gamma}$ a non self-dual Wadge class of Borel sets, $\mathbb{S}$ in $\mathbf{\Gamma}(\lceil T_d\rceil)$ not separable from $\lceil T_d\rceil\setminus\mathbb{S}$ by a pot($\check{\mathbf{\Gamma}}$) set, $(X_i)_{i\in d}$ a sequence of Polish spaces, and A_0, A_1 disjoint analytic subsets of $\Pi_{i\in d}\ X_i$. Then exactly one of the following holds:*

(a) The set A_0 is separable from A_1 by a pot($\check{\mathbf{\Gamma}}$) set.

(b) For each $i\in d$ there is $f_i:d^\omega\rightarrow X_i$ continuous such that the inequalities $\mathbb{S}\subseteq(\Pi_{i\in d}\ f_i)^{-1}(A_0)$ and $\lceil T_d\rceil\setminus\mathbb{S}\subseteq(\Pi_{i\in d}\ f_i)^{-1}(A_1)$ hold.

We now describe our significative example. The description of non self-dual Wadge classes due to Louveau and Saint Raymond is based on the following definition.

Definition 8 *Let $1\leq\xi<\omega_1$, $\mathbf{\Gamma}$ and $\mathbf{\Gamma}'$ two classes. Then*

$$A\in S_\xi(\mathbf{\Gamma},\mathbf{\Gamma}') \;\Leftrightarrow\; A=\bigcup_{n\geq 1}\ (A_n\cap C_n)\cup(B\setminus\bigcup_{n\geq 1}\ C_n)$$

for some sequence A_n in $\mathbf{\Gamma}$, $B\in\mathbf{\Gamma}'$, and a sequence $(C_n)_{n\geq 1}$ of pairwise disjoint $\mathbf{\Sigma}^0_\xi$ sets.

Example. We will study the following example: $\mathbf{\Gamma}:=S_2(\bigcup_{n\geq 1}\ \mathbf{\Pi}^0_{n+1},\mathbf{\Sigma}^0_3)$. Note that

$$\mathbf{\Sigma}^0_3=S_3(\{\check{\emptyset}\},\{\emptyset\}),$$

and similarly $\mathbf{\Pi}^0_{n+1}$ can be described with the operation S_ξ.

Notation. We will use the following topologies on $(\omega^\omega)^d$. Let X be a recursively presented Polish space. We denote by Δ_X the topology on X generated by $\Delta^1_1(X)$. Louveau proved that this topology is Polish.

• The topology τ_1 will be the product topology $\Delta^d_{\omega^\omega}$.

• Let $2\leq\xi<\omega_1^{\mathrm{CK}}$. The topology τ_ξ is generated by $\Sigma^1_1\big((\omega^\omega)^d\big)\cap\mathbf{\Pi}^0_{<\xi}(\tau_1)$.

We are ready to specify the Σ^1_1 set mentionned at the beginning of this talk. Before discussing our example, let us focus on the (relatively) simple case $\mathbf{\Gamma}=\mathbf{\Pi}^0_\xi$. Here, the Σ^1_1 set is $A_1\cap\overline{A_0}^{\tau_\xi}$. Its emptyness is equivalent to the fact that A_0 is separable from A_1 by a pot($\mathbf{\Pi}^0_\xi$) set. For our example $\mathbf{\Gamma}=S_2(\bigcup_{n\geq 1}\ \mathbf{\Pi}^0_{n+1},\mathbf{\Sigma}^0_3)$, the following result (which extends to the general case) holds. It strengthens Theorem 18.

Theorem 9 *Let T_d be a tree with Δ^1_1 suitable levels, $\mathbf{\Gamma}$ a non self-dual Wadge class of Borel sets, $\mathbb{S}\in\mathbf{\Gamma}(\lceil T_d\rceil)$ not separable from $\lceil T_d\rceil\setminus\mathbb{S}$ by a pot($\check{\mathbf{\Gamma}}$) set, and $A_0,A_1\in\Sigma^1_1\big((\omega^\omega)^d\big)$ disjoint. Then the following are equivalent:*

(a) The set A_0 is not separable from A_1 by a pot($\check{\mathbf{\Gamma}}$) set.

(b) The set A_0 is not separable from A_1 by a $\Delta^1_1\cap$pot($\check{\mathbf{\Gamma}}$) set.

(c) The set A_0 is not separable from A_1 by a $\check{\mathbf{\Gamma}}(\tau_1)$ set.

(d) $A_1\cap\overline{A_0\cap\bigcap_{n\geq 1}\overline{A_0\cap\overline{A_1}^{\tau_{n+1}}}^{\tau_2}}^{\tau_3}\neq\emptyset$.

(e) For each $i\in d$ there is $f_i:d^\omega\rightarrow\omega^\omega$ continuous such that the inequalities $\mathbb{S}\subseteq(\Pi_{i\in d}\ f_i)^{-1}(A_0)$ and $\lceil T_d\rceil\setminus\mathbb{S}\subseteq(\Pi_{i\in d}\ f_i)^{-1}(A_1)$ hold.