On minimal cofinalities

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Outline

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Definition

By ultrapower we mean the usual modeltheoretic ultrapower: $(\omega, <)^\omega/\mathscr{U}$ is the structure with domain $\{[f]_\mathscr{U} \mid f \in \omega^\omega\}$ where $[f]_\mathscr{U} = \{g \in \omega^\omega \mid \{n \mid f(n) = g(n)\} \in \mathscr{U}\}$ and $[f]_\mathscr{U} \leq_\mathscr{U} [g]_\mathscr{U}$ iff $\{n \mid f(n) \leq g(n)\} \in \mathscr{U}$. The minimal cofinality of an ultrapower of ω, $\mathfrak{mcf}$, is defined as the

$$\mathfrak{mcf} = \min\{\mathrm{cf}((\omega, <)^\omega)/\mathscr{U}) \mid \mathscr{U} \text{ non-principal ultrafilter on } \omega\}.$$

The cofinality of the symmetric group

Definition

$\mathrm{Sym}(\omega)$ is the group of all permutations of ω. If $\mathrm{Sym}(\omega) = \bigcup_{i<\kappa} G_i$ and $\langle G_i \mid i < \kappa \rangle$ is strictly increasing and each G_i is a proper subgroup of $\mathrm{Sym}(\omega)$ we call $\langle G_i \mid i < \kappa \rangle$ a decomposition. We call the minimal such κ the cofinality of the symmetric group, short $\mathrm{cf}(\mathrm{Sym}(\omega))$.

Outline

Groupwise dense families and ideals

A $\mathscr{G} \subseteq [\omega]^\omega$ is called a groupwise dense family if

- $\mathscr{G}$ is closed under almost subsets
- and for every strictly increasing sequence π_i, $i \in \omega$ there is $A \in [\omega]^\omega$ such that $\bigcup_{i\in A}[\pi_i, \pi_{i+1}) \in \mathscr{G}$.

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The groupwise density number, $\mathfrak{g}$, (groupwise density number for filters, $\mathfrak{g}_f$) is the minimal size of a collection of groupwise dense sets (ideals) whose intersection is empty.

Inequalities in ZFC

Observation

$\mathfrak{mcf} \geq \mathfrak{g}_f$.

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Proof: Let $X \subseteq \omega$ be infinite. The next function of X is:
$\nu_X(n) = \min(X \cap [n, \infty))$.
For $h\colon \omega \to \omega$ let

$$\mathscr{G}_h = \{X \subseteq \omega \mid h \leq_{\mathscr{U}} \nu_X\}.$$

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Theorem, Brendle and Losada, 2003

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Theorem, Shelah 2007

$\mathfrak{g}_f \leq \mathfrak{b}^+$ in ZFC.

A result about $\mathrm{cf}(\mathrm{Sym}(\omega))$ and $\mathfrak{mcf}$

Theorem, Banakh, Repovš, Zdomskyy

If there is no Q-point, then $\mathrm{cf}(\mathrm{Sym}(\omega)) \leq \mathfrak{mcf}$.

Theorem

Suppose that $\aleph_1 \leq \partial = \mathrm{cf}(\partial) \leq \theta = \mathrm{cf}(\theta) < \kappa = \mathrm{cf}(\kappa) < \lambda$ *and* GCH *holds up to* λ*. Then there is a notion of forcing* $\mathbb{P}$ *of size* λ *that preserves cardinalities and cofinalities and that forces* $\mathsf{MA}_{<\partial}$ *and* $\mathfrak{b} = \theta$ *and* $\mathfrak{mcf} \geq \kappa$ *and* $\mathfrak{c} = \lambda$*.*

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Outline

Almost Disjointness and a Square with Built-in Club Guessing

Hypothesis

GCH holds up to λ,
$\aleph_1 \leq \partial = \mathrm{cf}(\partial) \leq \theta = \mathrm{cf}(\theta) < \kappa = \mathrm{cf}(\kappa) < \lambda, \mu^+ = \lambda$.

Lemma

By a preliminary forcing of size λ that preserves cofinalities and cardinalities starting from the hypothesis we get a forcing extension with the following situation:

(a) $\partial = \mathrm{cf}(\partial) < \kappa = \mathrm{cf}(\kappa) \leq \mu < \lambda = \lambda^{<\lambda}$, $\mu^+ = \lambda$, $\mu^{\aleph_0} < \lambda$.

(b) $\mathscr{A}_\ell$ *is a family of subsets of* $[\mu]^{<\kappa}$,
$(\forall A \in \mathscr{A}_0)(\forall B \in \mathscr{A}_1)(A \cap B$ *is finite*$)$.

The Continuation of the Lemma

(c) if $\kappa_1 < \kappa$ and (u_0, u_1) is a partition of μ then there is $\ell \in 2$ and there a are λ sets $A \in \mathscr{A}_\ell$ such that $A \subseteq u_\ell$ and $|A| \geq \kappa_1$.

(d) there is a square sequence $\bar{C} = \langle C_\alpha \mid \alpha \in \lambda, \alpha \text{ limit}\rangle$ in $\lambda = \mu^+$ that is club guessing, i.e., $\bar{C}$ has the following properties

(1) $C_\alpha \subseteq \alpha$ is cofinal in α and closed in α, i.e., $\mathrm{acc}(C_\alpha) \subseteq C \cup \{\alpha\}$, $\mathrm{otp}(C_\alpha) \leq \mu$,

(2) for $\beta \in \mathrm{acc}(C_\alpha)$, $C_\beta = C_\alpha \cap \beta$,

(3) for every club E in λ there are stationarily many $\alpha \in \lambda$ with $\mathrm{cf}(\alpha) = \mu$ and $C_\alpha \subseteq E$. We call this "$\bar{C}$ is club guessing".

(e) There is an $\leq^*$-unbounded sequence $\langle g_\alpha \mid \alpha < \theta\rangle$ in ω^ω.

Outline

Redefining the Ground Model

Now we assume that we have families $\mathscr{A}_0$, $\mathscr{A}_1$ and a square sequence with built in club guessing $\bar{C}$ and an unbounded sequence $\langle g_\alpha \mid \alpha < \theta \rangle$ as described in Lemma in the ground model.

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Now we assume that we have families $\mathscr{A}_0$, $\mathscr{A}_1$ and a square sequence with built in club guessing $\bar{C}$ and an unbounded sequence $\langle g_\alpha \mid \alpha < \theta \rangle$ as described in Lemma in the ground model.
The final two forcing orders look like this:
The first step is a forcing $\mathbb{K} = (\mathbf{K}, \leq_{\mathbb{K}})$ of approximations $\mathbf{q} \in \mathbf{K}$, where $\mathbf{K} = \bigcup\{\mathbf{K}_\alpha \mid \alpha < \lambda\}$ and $\mathbf{K}_\alpha$ is the set of α-approximations. The relation $\leq_{\mathbb{K}}$ denotes prolonging the forcing iteration and taking an end extension of the partition of the iteration length and of $\bar{A}$. Once we have a generic $\mathbf{G}_{\mathbb{K}}$ for this forcing by approximations and end extension, we force with the direct limit

$$\mathbb{P}_{\underset{\sim}{\mathbf{G}}_{\mathbb{K}}} = \bigcup\{\mathbb{P}^{\mathbf{q}} \mid \mathbf{q} \in \underset{\sim}{\mathbf{G}}_{\mathbb{K}}\}.$$

Let $\alpha < \lambda$.

Definition

Assume that $\mathscr{A}_\ell$, $\ell = 0, 1$, λ, μ, κ, ∂, $\bar{g}$ and $\bar{C}$ have the properties listed above. A finite support iteration together with three disjoint domains and the sequence $\bar{A}$ of subsets of μ, $\mathbf{q} = (\mathbb{P}^{\mathbf{q}}, \mathscr{U}_0^{\mathbf{q}}, \mathscr{U}_1^{\mathbf{q}}, \mathscr{U}_2^{\mathbf{q}}, \bar{A})$, is an element of the set $\mathbf{K}_\alpha$ of *α-approximations* iff it has the following properties:

(a) $\mathbb{P}^{\mathbf{q}} = \mathbb{P}^{\mathbf{q}}_\alpha$, where $\bar{\mathbb{Q}}^{\mathbf{q}} = \langle \mathbb{P}^{\mathbf{q}}_\gamma, \underset{\sim}{\mathbb{Q}}^{\mathbf{q}}_\beta \mid \beta < \alpha(\mathbf{q}), \gamma \leq \alpha(\mathbf{q}) \rangle$ is a finite support iteration of c.c.c. forcings of length $\alpha(\mathbf{q}) = \lg(\mathbf{q}) < \lambda$.

(b) $\mathscr{U}_0 = \mathscr{U}_0^{\mathbf{q}}$ are the odd ordinals in α and $\mathscr{U}_1$, $\mathscr{U}_2$ is a partition of the even ordinals in α, $\mathscr{U}_2$ contains only limit ordinals, and $\bar{A} = \langle A_\beta \mid \beta \in \alpha \cap \mathscr{U}_2 \rangle$.

(c) For $\beta \in \mathscr{U}_0$, $\mathbb{Q}_\beta$ is the Cohen forcing $({}^{\omega>}2, \triangleleft)$ and we call the generic real $\underset{\sim}{\varrho}_\beta$.

(d) For $\beta \in \mathscr{U}_1$, $\underset{\sim}{\mathbb{Q}}_\beta$ is a c.c.c. forcing of size $\partial_\beta < \partial$.

Continuation II

(e) For $\beta \in \mathscr{U}_2$, we first fix a cardinal $\kappa_\beta < \kappa$. Then we have a sequence $\langle \xi_{\beta,i} \mid i < \kappa_\beta \rangle =: \bar{\xi}_\beta$ of $\xi_{\beta,i} = \xi(\beta, i) \in \mathscr{U}_0 \cap \beta$, increasing with i, of Cohen reals relevant for time β, and we have $t_\beta \in 2$ such that $\{\xi_{\beta,i} \mid i < \kappa_\beta\} \subseteq \{\varepsilon + 1 \mid \varepsilon \in \mathrm{acc}(C_\beta)\}$ and

$$\begin{aligned}&\big(A_\beta \in \mathscr{A}_{t_\beta} \wedge A_\beta \notin \{A_\gamma \mid \gamma \in \beta \cap \mathscr{U}_2\}\\ &\wedge A_\beta \supseteq \big\{\mathrm{otp}(\varepsilon \cap \mathrm{acc}(C_\beta)) \mid (\varepsilon \in \mathrm{acc}(C_\beta)\\ &\wedge \varepsilon + 1 \in \{\xi_{\beta,i} \mid i < \kappa_\beta\})\big\}\big).\end{aligned}$$

Continuation III

(f) For $\beta \in \mathscr{U}_2$ we define $\mathbb{P}_{\beta+1}$ as follows: First we have

($*_1$) a sequence $\bar{\underset{\sim}{\eta}}_\beta = \langle \underset{\sim}{\eta}_{\beta,i} \mid i < \kappa_\beta \rangle$, such that $\underset{\sim}{\eta}_{\beta,i}$ is a $\mathbb{P}_{\xi_{\beta,i}}$-name for an element of ω^ω.

($*_2$) $\bar{p}_\beta = \langle p_{\beta,i} \mid i < \kappa_\beta \rangle$, $p_{\beta,i} \in \mathbb{P}'_{\xi(\beta,i+1)}$, $\xi(\beta,i) \in \mathrm{dom}(p_{\beta,i})$.

We let $p \in \mathbb{P}_{\beta+1}$ iff $p \colon \beta + 1 \to \mathbf{V}$, $p \restriction \beta \in \mathbb{P}_\beta$ and

$$\begin{aligned} p \restriction \beta \Vdash_{\mathbb{P}_\beta} p(\beta) = (n, f, u) & \\ & \wedge n \in \omega \\ & \wedge f \colon n \to \omega \\ & \wedge u \subseteq \kappa_\beta \text{ is finite} \\ & \wedge (\forall i \in u)(p_{\beta,i} \in \mathbf{G}(\underset{\sim}{\mathbb{P}}_\beta)) \\ & \wedge |\{i \in \kappa_\beta \mid p_{\beta,i} \in \mathbf{G}(\underset{\sim}{\mathbb{P}}_\beta)\}| = \kappa_\beta \end{aligned}$$

$p \leq_{\mathbb{P}_{\beta+1}} q$ iff

$$\begin{aligned} q \restriction \beta \Vdash_{\mathbb{P}_\beta} n_{p(\beta)} \leq n_{q(\beta)} & \\ & \wedge f_{p(\beta)} \subseteq f_{q(\beta)} \\ & \wedge (\forall n \in [n_{p(\beta)}, n_{q(\beta)}))(\forall i \in u_{p(\beta)}) \\ & (\underset{\sim}{\varrho}_{\xi_{\beta,i}}(n) = t_\beta \to \underset{\sim}{\eta}_{\beta,i}(n) < f_{q(\beta)}(n)) \end{aligned}$$

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End of the list: Working towards compatibility

(g) For $\beta \leq \alpha$ we define $\mathbb{P}'_\beta$ to be the set of the $p \in \mathbb{P}_\beta$ with the following properties: If $\gamma \in \mathrm{dom}(p)$, then $p(\gamma) \in \mathbf{V}$ (is not just a name) and if $\gamma \in \mathrm{dom}(p) \cap \mathscr{U}_2$ then

$$\begin{aligned} p \restriction \gamma \Vdash i \in u_{p(\gamma)} \to \big(p_{\gamma,i} &\leq_{\mathbb{P}_\gamma} p \restriction \gamma \\ &\wedge \xi_{\gamma,i} \in \mathrm{dom}(p) \\ &\wedge p \restriction \xi_{\gamma,i} \text{ forces a value to } \underset{\sim}{\eta}_{\gamma,i} \restriction \lg(p(\xi_{\gamma,i})), \\ &\wedge n_{p(\gamma)} \leq \lg(p(\xi_{\gamma,i}))\big). \end{aligned}$$

Definition

We let $\mathbf{K} = \bigcup\{\mathbf{K}_\alpha \mid \alpha < \lambda\}$ be the set of approximations. For $\mathbf{q} = (\mathbb{P}_\alpha, \mathscr{U}_0, \mathscr{U}_1, \mathscr{U}_2, \bar{A}) \in \mathbf{K}_\alpha$ and $\beta < \alpha$ we let $\mathbf{q} \restriction \beta = (\mathbb{P}_\beta, \mathscr{U}_0 \cap \beta, \mathscr{U}_1 \cap \beta, \mathscr{U}_2 \cap \beta, \bar{A} \restriction \beta)$. We let the forcing with approximations be $\mathbb{K} = (\mathbf{K}, \leq_{\mathbb{K}})$ with the following forcing order: $\mathbf{q} \geq_{\mathbb{K}} \mathbf{q}_0$ iff $\mathbf{q} \restriction \alpha(\mathbf{q}_0) = \mathbf{q}_0$.

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Lemma

(1) For $\alpha < \lambda$, each $\mathbf{q} \in \mathbf{K}_\alpha$ has the c.c.c.

(2) If $\alpha < \lambda$ and $\mathbf{q} \in \mathbf{K}_\alpha$ and $\beta < \alpha$ then $\mathbf{q} \restriction \beta \in \mathbf{K}_\beta$.

Lemma

(1) $\mathbb{K} = (\mathbf{K}, \leq_{\mathbb{K}})$ *is a* $(< \lambda)$*-closed partial order.*

(2) $\Vdash_{\mathbb{K}} \mathbb{P}_{\underset{\sim}{\mathbf{G}}_{\mathbb{K}}}$ *satisfies the c.c.c.*

(3) Forcing by $\mathbb{K} * \mathbb{P}_{\underset{\sim}{\mathbf{G}}_{\mathbb{K}}}$ *does not collapse cofinalities nor cardinals and it forces* $2^{\aleph_0} = \lambda = \lambda^{<\lambda}$ *and the power* μ^{κ} *for* $\mu \geq \lambda$ *does not change.*

First properties of the overall order

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Lemma

In the generic extension by $\mathbb{P} = \mathbb{K} * \mathbb{P}_{\mathbf{G}_{\mathbb{K}}}$, $\mathsf{MA}_{<\partial}$ *holds and* $\mathfrak{mcf} \geq \kappa$.

Why Does $\bar{g}$ Stay Unbounded?

Lemma

If $\mathbf{q} \in K_\alpha$ and $\beta \leq \alpha$ then $\mathbb{P}'_\beta = (\mathbb{P}')^{\mathbf{q}}_\beta$ is a dense subset of $\mathbb{P}_\beta = \mathbb{P}^{\mathbf{q}}_\beta$.

Proof: Let for $\beta_1 < \beta_2 \leq \alpha$, $\mathbb{P}'_{\beta_1,\beta_2} = \{p \in \mathbb{P}_{\beta_2} \mid$ the demands from item (g) hold for $\gamma \in \mathrm{dom}(p) \smallsetminus \beta_1$ for all $i \in u_{p(\gamma)} \smallsetminus \beta_1$, and if $i \in u_{p(\gamma)} \cap \beta_1$ then we only demand $p_{\gamma,i} \leq p \restriction \gamma$ and $\xi_{\gamma,i} \in \mathrm{dom}(p)\}$

In Terms of Memory $\bar{g}$ is too Rich to be Dominated by a New Real

Lemma

Let $\bar{g} = \langle g_\varepsilon \mid \varepsilon < \theta \rangle$ be a $\leq^*$-increasing sequence in $\mathbf{V}$ that does not have an upper bound, $\partial \leq \theta < \kappa$. Then, for every $\alpha < \lambda$ and $\mathbf{q} \in \mathbf{K}_\alpha$, after forcing with $\mathbb{P}^{\mathbf{q}}$ the sequence $\bar{g}$ is still unbounded.

Corollary

After forcing with $\mathbb{P}$, $\bar{g}$ is unbounded.

Outline

Definition

(1) For $h \in \mathrm{Sym}(\omega)$, let $\mathrm{supp}(h) = \{n \mid h(n) \neq n\}$.

(2) For $u \subseteq \omega$ let $H_u = \{f \in \mathrm{Sym}(\omega) \mid \mathrm{supp}(f) \subseteq u\}$.

(3) Let $w_i = \{k \in \omega \mid k \equiv i \mod 3\}$.

(4) Let $u_i = \{k \in \omega \mid k \not\equiv i \mod 3\}$.

Definition

(1) We say $\bar{e}$ is a *witness for the decomposition* $\bar{G} = \langle G_i \mid i < \kappa \rangle$ iff $\bar{e} = \langle e_i \mid i < \kappa \rangle$ and $e_i \in G_{i+1} \setminus G_i$ and e_i is of order 2 and $e_i \in H_{w_1}$.

(2) $\bar{e}$ is a *witness* iff there is a decomposition $\bar{G}$ such that $\bar{e}$ is a witness for $\bar{G}$.

Witnesses for Decompositions

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(2) $\bar{e}$ is a *witness* iff there is a decomposition $\bar{G}$ such that $\bar{e}$ is a witness for $\bar{G}$.

Lemma

Every decomposition $\bar{G}$ such that all recursive permutations are in G_0 has a witness.

Outline

Conjugations by Conjugators of Order Two

Lemma

Suppose e, f *are permutations of order 2 and* $\mathrm{supp}(e) \subseteq w_1$ *and* $\mathrm{supp}(f) \subseteq w_0$ *and both supports are infinite. Then there is* g *of order 2,* $\mathrm{supp}(g) \subseteq u_2$ *such that*

$$e = g \circ f \circ g.$$

A Description of the Proof of Our Second Result in Words

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In the new kinds of iterands we add conjugators to get rid of short sequences of witnesses. This is similar to adding a dominator on a new ultrafilter set: We now add a new conjugator conjugating all members of a witness to one new function.